Collective modes for an array of magnetic dots with perpendicular magnetization

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The dispersion relations of collective oscillations of the magnetic moment of magnetic dots arranged in square-planar arrays and having magnetic moments perpendicular to the array plane are calculated. The presence of the external magnetic field perpendicular to the plane of array, as well as the uniaxial anisotropy for single dot are taken into account. The ferromagnetic state with all the magnetic moments parallel and chessboard antiferromagnetic state are considered. The dispersion relation yields information about the stability of different states of the array. There is a critical magnetic field below which the ferromagnetic state is unstable. The antiferromagnetic state is stable for small enough magnetic fields. Here the dispersion relations for collective modes for two phases, ferromagnetic and chessboard antiferromagnetic, within the whole Brillouin zone, are calculated. Nonstandard behavior of the mode frequencies on the wave vector is present for many cases. As the value of the wave vector approaches zero, for both phases a nonanalytic behavior of the mode frequency is found. For ferromagnetic state, the center of the Brillouin zone corresponds to a nonparabolic minimum, common to that is known for continuous thin films. For antiferromagnetic state, the saddle point with nonanalytic dependence of the components of the wave vector is located at small values of the wave vector. Nontrivial Van Hove anomalies are also found for both ferromagnetic and antiferromagnetic states.

I. INTRODUCTION

During the last decade the most impressive achievements in magnetism were related to fabrication, investigation, and application of artificial magnetic materials (see Ref. 1 for a recent review). The technologies of sputtering and lithography have progressed to the state where the manufacture of nanosize, periodic magnetic superlattices of different types is feasible. Among them two-dimensional (2D) lattices of submicron magnetic particles (so-called magnetic dots) attract much attention. These magnetic dots, in the form of circular or elliptic cylinders, or rectangular prisms, are made of soft magnetic materials such as Co and permalloy,[2–6] or highly anisotropic materials such as Dy[7] and FePt,[8] and the dot array lattice is usually designed to be quadratic or rectangular. In the array dots are separated from each other so that direct exchange interaction between dots is completely absent. Thus the dipolar interaction is the sole source of coupling between dots and determines the pattern of dot magnetic moment orientations that constitutes the physical properties of a dot array. Owing to the absence of exchange, magnetic dot arrays constitute promising material for high-density magnetic storage media. For this purposes, the dense arrays of small enough magnetic dots with the magnetic moments perpendicular to the array plane are optimal, see Refs. 9–11. For small enough dots (the diameter < 100 nm), the magnetization inside of a dot is almost uniform, producing the total magnetic moment $m_0 \gg \mu_B$, where μ_B is Bohr magneton. Therefore, this can be a new kind of magnetic material with purely two-dimensional lattice structure and pure dipolar coupling between large enough magnetic moments.

To describe the physical properties of magnets, in general, it is essential to take into account interactions having various origins and different energy scales. For adjacent spins the spin exchange interaction is almost invariably the strongest. If this interaction is ferromagnetic it causes uniform spin ordering over distances that usually substantially exceeds the atomic spacing. The dipole-dipole interaction, though as a rule is considerably weaker than the exchange for adjacent spins, but it extends to much longer range. The competition of these two interactions produces magnetic domain structure with long-range nonuniformity of magnetization for usual magnetic samples.

Models of magnetic moments with the dipolar interaction have been theoretically studied for more than 60 years, and many physical properties, lacking in the spin-exchanged systems, are known for those models.[12,13] Note first the presence of a nonunique ground state with nontrivial continuous degeneracy that is typical both for three-dimensional (3D) lattices[13] and for two-dimensional lattices of planar[14,15] and three-dimensional dipoles.[16] Two-dimensional systems of Ising-type dipoles with perpendicular anisotropy demonstrate a complicated behavior when subject to an external magnetic field perpendicular to the array's plane.[17] At zero field, the ground state of a square 2D array is chessboard antiferromagnetic (AFM), and this state is stable at low fields $H < H_1 = 2.3M$, where $M = m_0/a^3$ is characteristic field of magnetic-dipole interaction and a is the lattice spacing. The ferromagnetic (FM) (saturated) state is stable at high fields $H > H_0 = 9.03M$ with a cascade of phase transitions between complicated magnetic structures at $H_1 < H < H_0$.

The models[12–14] were discussed originally in regard to the description of real crystalline spin systems in which the dipolar interaction is predominant, such as some rare-earth magnets. However, fabrication and experimental study of magnetic dot arrays provide new physical systems for the testing of basic magnetism models. For many materials, such

as compounds with rare-earth ions, granular magnets, and diluted solid solution of paramagnetic ions in nonmagnetic crystals, the magnetic properties seem to be similar to those of dot arrays. Nevertheless, magnetic dot lattices exhibit physical properties, which are absent in all above-mentioned systems. First, dot arrays in contrast to layered crystals are literally two dimensional. The manifestations of long-ranged magnetic-dipole interaction are principally different for of layered crystals and truly two-dimensional systems.[18] Second, the scale of dipolar interaction of two spins $(2\mu_B S)^2/a^3$, where μ_B the Bohr magneton, a is the interspin distance, even with large spins like $S=7/2$ does not exceed several Kelvin, whereas for dots with volume as small as 10^4 nm^3 the characteristic magnetic moment $m_0 > 10^3 \mu_B S$ and the characteristic energy can be comparable to or even higher than the thermal energy at room temperature. Besides, for a compound with high density of rare-earth ions though the exchange interaction is small it is not completely negligible.

As both FM and AFM exchange interactions of adjacent spins lead to magnetic states essentially different from those caused by dipolar interaction, these systems cannot be considered as purely dipolar. The large magnetic moment of each particle is typical also for granular materials, but the magnetic dot lattices are different from the latter by the high spatial regularity. It is worth mentioning a new class of materials—molecular crystals with high-spin molecules.[19] However, they are 3D, and the spin of these systems is much less than the effective spin for magnetic dots not exceeding 10 or 15. It is also important that the size of magnetoactive part of a single high-spin molecule is small compared to the total size of molecule; therefore the dipole interaction is weak.

Hence, magnetic dot arrays are specific new magnetic materials with purely 2D and quite regular lattice structure and long-distance dipolar coupling between magnetic moments, which are rather large and manifest at high temperature. From the point of view of dynamical properties this implies that for magnetic dot arrays the well-defined modes of collective oscillations characterized by definite quasimomentum should exist, while this is admittedly not the case for granular magnets or dilute solid solutions of paramagnetic ions. The direct measurement of the dependence $\omega(\vec{k})$ can be done by the Brillouin light-scattering method. In the pioneering experiments[20–22] no indication for a band structure due to the periodic arrangement of the dots was found, see also review articles.[23,24] However, more recently the dispersion effects for dense arrays were clearly observed by the Brillouin light-scattering method[25–27] and time-resolved scanning Kerr microscopy,[28] stimulating the development of the theory for these systems.

In first theoretical articles, the finite systems were investigated, sometimes with large enough number of dots N, such as $N \sim 10^3 - 10^4$, see Ref. [29] or even smaller systems.[26] The analytical calculations were performed using the Bloch theorem which is applicable to infinite arrays. The cases of spherical particles,[30–32] cylindrical particles in uniform state,[33–35] and in the vortex state,[36] were also considered, see for review Ref. [37].

In the present work we considered collective modes for a square lattice of rather small dots of nearly ellipsoidal shape

in which magnetization can be considered as uniform within a single dot in the presence of an external bias magnetic field. Only the simplest magnetic states of the magnetic dot system, namely, FM state and chessboard AFM state are considered.

The long-range character of interaction of magnetic dots leads to unique properties of collective excitations in this system. Especially for the FM state nonanalytic dependence of collective mode frequencies on quasimomentum $\vec{k}$, known for thin continuous magnetic films, appears. Namely, as $\vec{k} \to 0$ the spectrum has a finite gap ω_0, but the dispersion law is nonanalytic $\omega \to \omega_0 + c|\vec{k}|$. For AFM state, the spectrum consists from two energy bands, which are connected at the border of the Brillouin zone of the lattice at zero magnetic field, but these two bands are well separated by an energy gap at finite fields. For AFM state, the singularity described above appears at nonsmall values of wave vector. Instead, near the center of the Brillouin zone, the unusual extremum, which is a saddle point in the center of the Brillouin zone, is found. The spectra for both FM and AFM phases are very sensitive to the value of external magnetic field that open the possibility to create field-operated magnonic materials.

II. MODEL DESCRIPTION

For uniform magnetization inside of each particle the state of the dot array is described by the full magnetic moment of each dot, $\vec{m}_{\vec{l}}$, $|\vec{m}_{\vec{l}}| = m_0$ placed in the square-lattice sites $\vec{l} = (\vec{a}_1 l_1 + \vec{a}_2 l_2)$, where $\vec{a}_1 = a\vec{e}_x$ and $\vec{a}_2 = a\vec{e}_y$ are the elementary translation vectors, l_1, l_2 are integers. The system Hamiltonian describing the dipole interaction of dots subject to an external magnetic field $\vec{H}_0$ and taking into account the uniaxial magnetic anisotropy for each dot reads[38]

$$\hat{H} = \frac{1}{2}\sum_{\vec{l} \neq \vec{l}'} \frac{\vec{m}_{\vec{l}}\vec{m}_{\vec{l}'} - 3(\vec{m}_{\vec{l}}\vec{\nu})(\vec{m}_{\vec{l}'}\vec{\nu}')}{|\vec{l} - \vec{l}'|^3} - \sum_{\vec{l}} \left[\frac{1}{2}\kappa \cdot (\vec{m}_{\vec{l}}\vec{e}_z)^2 + \vec{m}_{\vec{l}}\vec{H}_0 \right].$$

(1)

Here $\vec{\nu} = (\vec{l} - \vec{l}')/|\vec{l} - \vec{l}'|$, κ is the anisotropy constant for a given dot, which is assumed to be uniaxial with a easy axis $\vec{e}_z$, perpendicular to the dot array plane. For dots made of soft magnetic material, the anisotropy is associated with the dot shape ($\kappa > 0$ or $\kappa < 0$ for dots oblate or oblong along $\vec{e}_z$, respectively). For dots made of highly anisotropic magnetic materials,[7,8] the crystalline anisotropy of the material can provide some contribution to κ. The state of a single dot can be characterized by perpendicular magnetization ($\vec{m}_z = \pm m_0\vec{e}_z$). However, in the following only the most symmetric case will be considered where the external field $\vec{H}_0$ is perpendicular to the array plane.

Although oscillations of magnetization in the dot can be treated in a purely classical manner, it is convenient to employ the operator approach.[36] For dots with homogeneous magnetization it is natural to introduce magnon creation and annihilation operators for the total magnetic moment. To find a dispersion law in the linear approximation, it is sufficient to use the formulae,[38]

$$m_{3,\vec{l}} = m_0 - g\mu_B a_{\vec{l}}^\dagger a_{\vec{l}},$$

$$m_{1,\vec{l}} = (a_{\vec{l}}^\dagger + a_{\vec{l}})\sqrt{gm_0\mu_B/2},$$

$$m_{2,\vec{l}} = i(a_{\vec{l}}^\dagger - a_{\vec{l}})\sqrt{gm_0\mu_B/2}, \qquad (2)$$

below we will put Lande factor $g=2$, which in this approximation are similar to the familiar Holstein-Primakoff or Dyson-Maleev representations.[38] Here μ_B is the Bohr magneton, 1, 2, and 3 denote the projections on the set of orts, for example, $\vec{e}_x$, $\vec{e}_y$, and $\vec{e}_z$, for $\vec{m}=m_0\vec{e}_z$ or $\vec{e}_x$, $-\vec{e}_y$, and $-\vec{e}_z$ for $\vec{m}=-m_0\vec{e}_z$.

III. FERROMAGNETIC STATE OF ARRAY

For the FM state of the dot array ($\vec{m}=\vec{e}_z$ for all dots) the same set of orts in Eq. (2), $\vec{e}_1=\vec{e}_x$, $\vec{e}_2=\vec{e}_y$, and $\vec{e}_3=\vec{e}_z$ should be used. In the quadratic approximation over the operators $a_{\vec{l}}^\dagger$ and $a_{\vec{l}}$ the Hamiltonian reads

$$\hat{H} = 2\mu_B \sum_{\vec{l}} \left[\left(H_0 + H_a - \sum_{\vec{\delta}\neq 0} \frac{M}{|\vec{\delta}|^3} \right) a_{\vec{l}}^\dagger a_{\vec{l}} - \frac{1}{2} \sum_{\vec{\delta}\neq 0} \frac{M}{|\vec{\delta}|^3} a_{\vec{l}}^\dagger a_{\vec{l}+a\vec{\delta}} \right]$$
$$- \frac{3\mu_B M}{2} \sum_{\vec{l}} \left[\sum_{\vec{\delta}\neq 0} \frac{(\delta_x + i\delta_y)^2}{|\vec{\delta}|^5} a_{\vec{l}}^\dagger a_{\vec{l}+a\vec{\delta}}^\dagger + \text{H.c.} \right], \qquad (3)$$

where $\vec{\delta}=l_1\vec{e}_x + l_2\vec{e}_y$ is a dimensionless lattice vector, $H_a = \kappa m_0$ is the anisotropy field, and $M=m_0/a^3$ is the characteristic value defining the dipolar interaction intensity and having the same dimension as usual the 3D magnetization. The collective modes are introduced via states a_k and $a_k^\dagger$ of definite quasimomentum $\vec{k}$

$$a_k = \frac{1}{\sqrt{N}} \sum_{\vec{l}'} a_{\vec{l}} e^{i\vec{k}\vec{l}}, \quad a_k^\dagger = \frac{1}{\sqrt{N}} \sum_{\vec{l}} a_{\vec{l}}^\dagger e^{-i\vec{k}\vec{l}}, \qquad (4)$$

where N is the total number of dots in an array. The collective modes are defined by the quadratic Hamiltonian over a_k and $a_k^\dagger$ which acquires the standard form

$$\hat{H} = 2\mu_B M \sum_k \left[A_k a_k^\dagger a_k + \frac{1}{2}(B_k a_k^\dagger a_{-k}^\dagger + B_k^* a_k a_{-k}) \right]. \qquad (5)$$

When the form of coefficients A_k and B_k are established, the collective excitation energy $\varepsilon(\vec{k})=\hbar\omega(\vec{k})$ may be found by means of $u-v$ Bogolyubov transformations (see, for example, Ref. 38) and universally reads

$$\varepsilon(\vec{k}) = 2\mu_B M \sqrt{A_k^2 - |B_k|^2}, \quad \omega(\vec{k}) = \gamma M \sqrt{A_k^2 - |B_k|^2},$$

where $\gamma=2\mu_B/\hbar$ is the gyromagnetic ratio. The concrete forms of A_k and B_k are defined by the distribution of the magnetic moments within the array. For the case of interest (parallel ordering of dot magnetization) one can find

$$A_k = h + \beta - \frac{3}{2}\sigma(0) + \frac{1}{2}[\sigma(0) - \sigma(\vec{k})], \quad B_k = 3\sigma_c(\vec{k}), \quad (6)$$

where $h=H_0/M$ and $\beta=H_a/M$ are dimensionless magnetic field and anisotropy constant, respectively; and the dipolar sums $\sigma(\vec{k})$ and $\sigma_c(\vec{k})$ appear

$$\sigma(\vec{k}) = \sum_{\vec{l}\neq 0} \frac{1}{(l_1^2 + l_2^2)^{3/2}} e^{i\vec{k}\vec{l}},$$

$$\sigma_c(\vec{k}) = \sigma'(\vec{k}) + i\sigma''(\vec{k}) = \sum_{\vec{l}\neq 0} \frac{(l_1 - il_2)^2}{(l_1^2 + l_2^2)^{3/2}} e^{i\vec{k}\vec{l}}. \qquad (7)$$

Such sums naturally emerge for any problem involving interaction of dipoles ordered in a lattice. For the two-dimensional case the sums $\sigma(\vec{k})$ and $\sigma_c(\vec{k})$ have a series of peculiarities discussed in Refs. 33, 34, and 36. Now we note only that the sum $\sigma(\vec{k})$ at $\vec{k}=0$ converges for large $\vec{l}$ faster than in the three-dimensional case and has the finite value $\sigma(0)=9.03362$. On the other hand, the representation $\sigma(\vec{k})$ through the integral applicable in the three-dimensional case (see Ref. 38) is not feasible since the corresponding two-dimensional integral $\int dxdy/(x^2+y^2)^{3/2}$ diverges as x, $y\rightarrow 0$. Therefore, the regular presentation of the static demagnetization field $\vec{H}_m$ through the magnetization components $\vec{M}$ of the form $\vec{H}_m=-4\pi\sum_{i=1}^{3}(\vec{e}_i N_i M_i)$, where N_i are demagnetizing factors, $\sum_{i=1}^{3}N_i=1$, fails in the two-dimensional case. In particular, Yafet and Georgy demonstrated[39] that for the atomic monolayer the z projection of static demagnetizing field $\vec{H}_m$ is $(3/2)\sigma(0)$ in our notation, and it does not coincide with the continuum theory result for a thin film, $\vec{H}_m=-4\pi\vec{e}_z M_z$. The numerically obtained value of $\sigma(0)=9.03362$ gives $(3/2)\sigma(0)=1.0783\times 4\pi$, thus the difference is nearly 8%. For $\vec{k}\neq 0$ the specific feature of the two-dimensional case manifests itself more vividly as the appearance of a nonanalytical dependence on $\vec{k}$, namely, at $k\rightarrow 0$,

$$\sigma(\vec{k}) = \sigma(0) - k \cdot F(\vec{k}), \quad \sigma_c(\vec{k}) = \frac{(k_x - ik_y)^2}{\sqrt{k_x^2 + k_y^2}} \cdot G(\vec{k}), \quad (8)$$

where $k=|\vec{k}|$, $G(\vec{k})$ and $F(\vec{k})$ are the real functions, analytic as $k\rightarrow 0$ with the limit values $F(\vec{k})\rightarrow 2\pi a$, $G(\vec{k})\rightarrow 2\pi a/3$ as $\vec{k}\rightarrow 0$ and it is invariant under symmetry transformations of the square lattice. Thus in the collective oscillation spectra nonanalytic features appear which are absent in the case of a three-dimensional ferromagnet. From here it is seen that in contrast to $\sigma(\vec{k})$ the complex sum $\sigma_c(\vec{k})$ is not invariant with respect to symmetry transformations of the square lattice, but these transformations change only the phase factor, and therefore do not have an effect on the frequency. The frequency contains only $|\sigma_c(\vec{k})|=k\cdot G(\vec{k})$,

$$\omega(\vec{k}) = \gamma \sqrt{H_a^{\text{eff}} + H_0 + \frac{M}{2}\Sigma_{(+)}(\vec{k})} \cdot \sqrt{H_a^{\text{eff}} + H_0 + \frac{M}{2}\Sigma_{(-)}(\vec{k})}, \qquad (9)$$

where we introduced the combinations of dipole sums $\Sigma_{(\pm)}(\vec{k})$

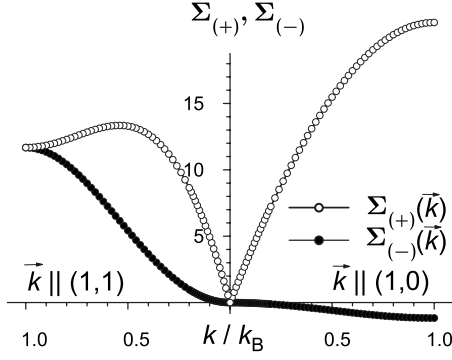$$\Sigma_{(\pm)}(\vec{k}) = \sigma(0) - \sigma(\vec{k}) \pm 3|\sigma_c(\vec{k})|. \qquad (10)$$

FIG. 1. The dependence of the combinations of the dipole sums $\Sigma_{(\pm)}(\vec{k})$ on the quasimomentum for symmetric directions of the square lattice, full symbols depict $\Sigma_{(-)}(\vec{k}) = \sigma(0) - \sigma(\vec{k}) - 3|\sigma_c(\vec{k})|$, open circles represent $\Sigma_{(+)}(\vec{k}) = \sigma(0) - \sigma(\vec{k}) + 3|\sigma_c(\vec{k})| \cdot k_B$ is the maximal value of the wave-vector modulus for a given direction, corresponding to the border of the Brillouin zone, $k_B = \pi/a$ for $\vec{k} \| (0,1)$ or $\vec{k} \| (1,0)$ and $k_B = \pi\sqrt{2}/a$ for $\vec{k} \| (1,1)$.

These sums are plotted in Fig. 1 as a function of $\vec{k}$ for symmetrical directions of the reciprocal lattice. The field, $H_a^{\text{eff}} = H_a - (3/2) M \sigma(0)$ is the z projection of the effective anisotropy field, comprising an anisotropy field for the single particle and demagnetization field of the array as a whole, and $\sigma(0)(3m_0/2a^3) \cong 1.08 \times 4\pi M$.

A. Dispersion relation and stability conditions

It is remarked that, the expression for the frequency can be rewritten in the universal form

$$\omega^2(\vec{k}) = [\omega_0 + \omega_{\text{int}}\Sigma_{(+)}(\vec{k})][\omega_0 + \omega_{\text{int}}\Sigma_{(-)}(\vec{k})] \qquad (11)$$

with the parameters $\omega_0 = \gamma(H_a^{\text{eff}} + H_0)$, $\omega_{\text{int}} = \omega_M L R^2/8a^3$, where ω_0 can be thought of as the mode-gap frequency, ω_{int} determines the magnitude of the mode dispersion caused by interaction, and $\omega_M = 4\pi\gamma M_s$ is a characteristic frequency of the material, which is $\omega_M = 30$ GHz for permalloy. It is worth noting, the same structure [Eq. (11)] appears for the collective mode of vortex precession for the array of vortex-state dots.[36] To define the parameters in Eq. (11) we used $\omega_{\text{int}} = \gamma M/2 = \gamma m_0/2a^3$ and the value of the magnetic moment $m_0 = \pi M_s L R^2$ for the single dot of cylindrical shape of thickness L and radius R.

Let us discuss the characteristic features of the dispersion relation represented by Eq. (11) in more detail. For small values of $|\vec{k}|$ both $\sigma(0) - \sigma(\vec{k})$ and $|\sigma_c(\vec{k})|$ are linearly increasing functions of $|\vec{k}|$ for all directions of $\vec{k}$, $\sigma(0) - \sigma(\vec{k}) \to 2\pi a|\vec{k}|$, $|\sigma_c(\vec{k})| \to 2\pi a|\vec{k}|/3$, as explained in the appendix. On the other hand the linear components in k compensate each other in the second bracket as $\vec{k} \to 0$ and only the quadratic terms in k remain (see Fig. 1). Because of this the first bracket in Eq. (11) in the long-wave limit acquires the form $H_{\text{tot}}^{\text{eff}} + 4\pi a k M$, $k = |\vec{k}|$. Thus the magnon spectra have a peculiarity as $\vec{k} \to 0$, $\omega(\vec{k}) \cong \omega_0 + 2\pi\gamma Ma|\vec{k}|$ (see Fig. 2). If one introduce an "effective surface magnetization" $M_s^{(surf)} = m_0/a^2$, having a sense of the magnetic moment per unit
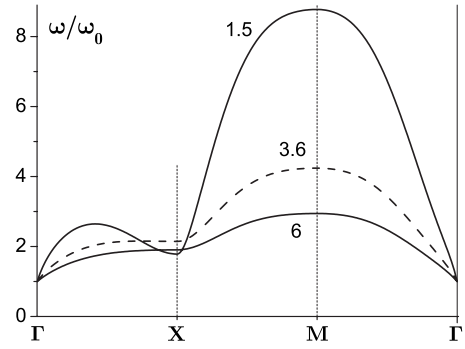

FIG. 2. The dispersion law for a dot array in the FM state along some symmetric directions at different $\lambda = \omega_0/\omega_{\text{int}}$ (shown near the curves). Here and below we use the notations Γ, X, and M for symmetric points of the Brillouin zone $(0,0)$, $(0,1)$, and $(1,1)$, respectively.

area of array, the coefficient of $|\vec{k}|$ is exactly the same as for a thin continuum magnetic film of saturation magnetization, M, and thickness, L, see Refs. 40 and 41, where it is natural to introduce $M_s^{(surf)} = M_s L$.

Since the spectrum's behavior depends on the value of the external magnetic field by a simple additive way, the sole parameter, determining the form of the spectrum, is the ratio $\lambda = \omega_0/\omega_{\text{int}}$. First, this dispersion relation is strongly anisotropic. For all values of $|\vec{k}|$ inside the Brillouin zone the frequency $\omega(\vec{k})$ is increasing monotonically for $\vec{k}$ parallel to the $(1,1)$ direction, but for $\vec{k}$ along the $(1,0)$ axis the dependence can be nonmonotonic, as seen in Fig. 2. The oscillations have $\omega^2 > 0$ and the ferromagnetic state is stable for weak enough interaction, $\omega_0 > 1.172\omega_{\text{int}}$. Near the point of instability the dependence $\omega(\vec{k})$ for $\vec{k} \| (1,0)$ has a minimum near the boundary of the Brillouin zone, which is present until $\lambda = 3.6$ which is also seen in Fig. 2. In the region of parameter space $1.172 < \omega_0/\omega_{\text{int}} < 3.6$ some more extrema and saddle points are present inside the Brillouin zone. For weaker interactions, $\omega_{\text{int}} < \omega_0/3.6$, the dependence of $\omega(\vec{k})$ becomes monotonic inside all of the Brillouin zone.

Here it is remarked that the stability condition for the dot array against small perturbations is nothing but the condition of positive definiteness of the function $\omega^2(\vec{k})$. The values of the sums $\Sigma_{(\pm)}(\vec{k})$ vanish as $k \to 0$, and the stability of the FM state against the linear *long-wave* excitations is broken if $H_a^{\text{eff}} + H_0 < 0$ or $H_0 < 3\sigma(0)/2 - H_a$. After the replacement $3\sigma(0)/2 \to 4\pi$ and $M \to M_s$ this criterion coincides with the result obtained from the continuum theory for a thin magnetic film. Nevertheless the dependence $\omega(\vec{k})$ is essentially different than for thin films. The minimal value of the function $\omega^2(\vec{k})$ is attained at the boundary of the Brillouin zone for the quasimomentum, $\vec{k}$ parallel to the $(1,0)$ axis. At this point, the maximal value of

$$-(1/2)\Sigma_{(-)}(\vec{k}) = 0.5[3|\sigma_c(\vec{k})| - \sigma(0) + \sigma(\vec{k})]$$

is equal to $0.5859 = 4\pi \times 0.0466$. Because this instability condition is first realized for a nonsmall $\vec{k} \| (1,0)$, it is stricter than the one found in the long-wave approximation. Finally,

the ferromagnetic state of the dot array loses stability at $H_0 < H_c$, where

$$H_c = \frac{M}{2} \max[3|\sigma_c(\vec{k})| - \sigma(0) + \sigma(\vec{k})] - H_a$$

$$\simeq 1.125 \times 4\pi M - H_a. \qquad (12)$$

This value of the critical field differs from that obtained by the continuum approximation, $4\pi M$ by more than 12%. Moreover, the character of instability for the array of dots coupled by the dipole interaction is principally different that for the continuous film. For the dot array as $H_0 \leq H_c$ the unstable mode has the maximal increment at $\vec{k} = (\pi/a)\vec{e}_x$ or $\vec{k} = (\pi/a)\vec{e}_y$ corresponding to the transition from the FM state with parallel magnetic moments for all the dots to the chessboard AFM state. For continuous films the instability occurs only for long-wave excitations and leads to appearance of long-period domain structures.[42]

B. Density of states

The complicated behavior of the dispersion curves plays an important role in the formation of Van Hove singularities. The Van Hove singularities are connected with the extrema of the dispersion law of quasiparticles, namely, with minima, maxima and saddle points. Any branch of collective excitations has at least one of these extrema within the Brillouin zone. For the interaction of a finite number of neighboring spins, the dispersion law is described by an analytical function, and the function $\omega(\vec{k})$ can be approximated by parabolic functions in the vicinity of an extremum. In this case the Van Hove singularities have a standard form. In particular, for a two-dimensional case the points of minima and maxima of $\omega(\vec{k})$, where $\omega = \omega_{min}$ or $\omega = \omega_{max}$, result in a finite jump of the density of states, $D(\omega) = C \cdot \Theta(\omega - \omega_{min})$ or $D(\omega) = C \cdot \Theta(\omega_{max} - \omega)$, respectively, where $\Theta(x)$ is the Heaviside step function, $\Theta(x) = 1$ for $x > 0$ and $\Theta(x) = 0$ for $x < 0$. Saddle points with $\omega = \omega_c$ lead to logarithmic singularities of the form $\Delta D(\omega) = C \ln[\omega_c/|\omega - \omega_c|]$, and the appearance or disappearance of one more singularity of such form at some value of frequency has to be clearly seen.

In our case the structure can be richer due to the long-range character of the interaction, as extrema can correspond to a nonstandard behavior of $D(\omega)$. First, note, the nonstandard (linear in k) dispersion near the gap frequency, ω_0, produce much weaker singularities in the density of states, for $\omega > \omega_0$ simple calculations give

$$D(\omega) = C(\omega - \omega_0) \cdot \Theta(\omega - \omega_0) \qquad (13)$$

instead of finite jump. In addition, the number of extrema could be more than three. In our case all these possibilities can be realized at various values of the parameter $\lambda = \omega_0/\omega_{int}$.

The analysis of the spectrum shows that for the FM state the dependence $\omega = \omega(\vec{k})$ always has a minimum in the point Γ ($\vec{k} = 0$); moreover, in the vicinity of this point $(\omega - \omega_0) \propto |\vec{k}|$. As well $\omega(\vec{k})$ always has the standard parabolic maximum at the point M $[\vec{k} = (\pi/a)(1,1)]$. Therefore, near upper

and lower edges of the frequency band, the character of the singularities of the density of states $D(\omega)$ is universal, $D(\omega) \propto (\omega - \omega_0)$ near the ω_0 [see Eq. (13)], and $D(\omega)$ has a finite jump near the maximal frequency, ω_{max}. At all values of λ, the logarithmic singularity of the form of $\Delta D(\omega) = C \ln[\omega_c/|\omega - \omega_c|]$ is also present (see Fig. 3).

For large λ, corresponding to a weak interaction of particles, the frequency grows with $|\vec{k}|$ for all directions of $\vec{k}$. In this case the situation is standard: saddle points are located at four symmetrical points of the type of X $(1,0)$, and only the three aforementioned singularities are present in the density of state function, see Fig. 3(a). However, for small λ, $\lambda < \lambda_c = 3.6$ the dependence $\omega(\vec{k})$ for $\vec{k} \| (0,1)$ is nonmonotonous: a local minimum with a standard parabolic dependence is placed in the points of the type of X $(1,0)$, where $\vec{k} = \pm \pi \vec{e}_x/a$ or $\vec{k} = \pm \pi \vec{e}_y/a$. Thus for small λ four saddle points are symmetrically located inside the Brillouin zone on the directions such as $(0,1)$ with $|\vec{k}| < \pi/a$. A local minimum leads to existence of an additional jump of the density of states having the form $D(\omega) = C_1 + C_2 \cdot \Theta(\omega - \omega_{lm})$ inside the frequency band for $\lambda < \lambda_c$, see Fig. 3(b), whereas the saddle points produce standard logarithmic singularities.

It is also seen that at the critical value of the parameter $\lambda = \lambda_c \simeq 3.6$, where two Van Hove singularities emerge at some critical value, ω_c, the density of states shows a new type of singularity

$$D(\omega) = \frac{C}{|\omega - \omega_c|^{1/4}}, \qquad (14)$$

which is stronger than the standard logarithmic singularity [compare Figs. 3(a)–3(c)]. To calculate this dependence, one have to take into account the terms of order $(\vec{k} - \vec{k}_0)^4$. It is remarked here that the value of the parameter $\lambda = \omega_0/\omega_{int}$ depends on the external magnetic field and can be changed continuously for the same sample during the experiment, thus, the observation of this singularity is possible. On the other hand, linear in k dispersion near the gap frequency ω_0 produces much weaker singularities in the density of states. In particular, for $\omega \geq \omega_0$ simple calculations give $\Delta D(\omega) = C \cdot (\omega - \omega_0)\Theta(\omega - \omega_0)$ instead of finite jump. This singularity is clearly seen for all values of λ (see Fig. 3).

IV. CHESSBOARD ANTIFERROMAGNETIC STATE

In the previous section we investigated the states of the array in which all dots are in the same magnetic state (FM ground state of array). Next consider the AFM structure where the transition from FM to AFM states can be controlled by a weak external magnetic field. The spectral analysis of the dot array in the AFM state is of much interest because this transition from FM to AFM states provides the possibility to tune the properties of collective modes in the system.

The method developed here might be easily generalized for the case of simple AFM states, which might be described within the framework of a few sublattices. To do that, it is necessary to split the array onto different sublattices with the same dot state in each sublattice and introduce different Bose operators for each sublattice.
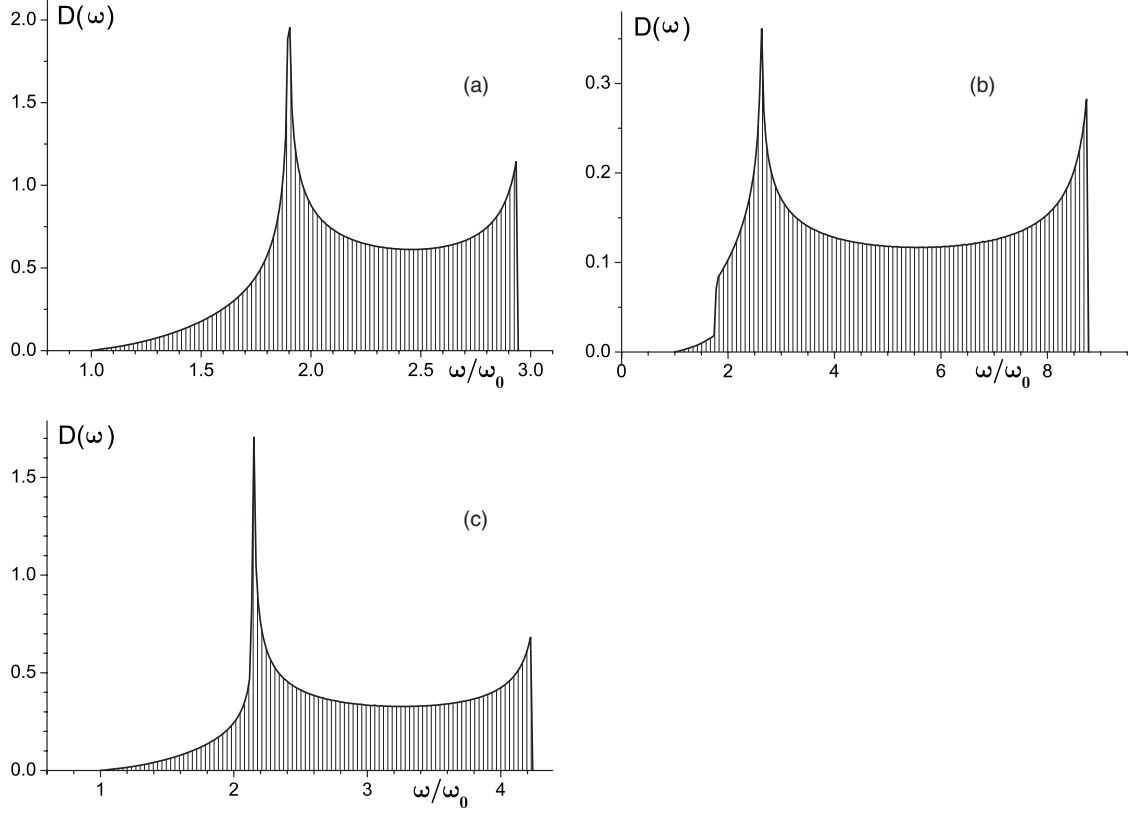
FIG. 3. The profile of the density of state function, normalized by the condition $\int D(\omega) d\omega / \omega_0 = 1$, for different values of the parameter $\lambda = \omega_0 / \omega_{int}$. (a) weak interaction, $\lambda = 6 > \lambda_c$; (b) strong interaction, $\lambda = 1.5 < \lambda_c$; and (c) special case $\lambda = \lambda_c$, where more strong singularity appears.

A. Brillouin zone, quasimomentum, and magnon operators

Let us consider the simplest chessboard antiferromagnetic state with the magnetic moments $\vec{m}_{\vec{l}} = (-1)^{l_1 + l_2} m_0 \vec{e}_z$ in the ground state. This system can be treated as the embedding of two quadratic sublattices with the lattice spacing $a\sqrt{2}$, as shown in Fig. 4. Let us denote the sites of the first sublattice which have $\vec{m}_{\vec{\delta}} = \vec{e}_z$ as $a\vec{\delta}$, $\vec{\delta} = \sqrt{2}(\delta_1 \vec{\varepsilon}_1 + \delta_2 \vec{\varepsilon}_2)$, where δ_1 and δ_2 are integers and $\vec{\varepsilon}_1 = (\vec{e}_x - \vec{e}_y)/\sqrt{2}$, $\vec{\varepsilon}_2 = (\vec{e}_x + \vec{e}_y)/\sqrt{2}$ are basis vectors for sublattices. Using orts $\vec{e}_x$, $\vec{e}_y$, the vector $\vec{\delta}$ can be expressed as $\vec{\delta} = \delta_x \vec{e}_x + \delta_y \vec{e}_y$, where $\delta_x = \delta_1 + \delta_2$, $\delta_y = \delta_2 - \delta_1$. The sites for second sublattice which have $\vec{m}_{\vec{\mu}} = -m_0 \vec{e}_z$ can be expressed as $a\vec{\mu}$, $\vec{\mu} = \sqrt{2}[(\mu_1 + 1/2)\vec{\varepsilon}_1 + (\mu_2 + 1/2)\vec{\varepsilon}_2]$ or $\vec{\mu} = \mu_x \vec{e}_x + \mu_y \vec{e}_y$, $\mu_x = \mu_1 + \mu_2 + 1$, $\mu_y = \mu_2 - \mu_1$, where μ_1 and μ_2 are integers.

To describe the magnetic oscillations for two sublattices, it is sufficient to use two Bose operators as follows:

$$m_{z,\vec{\delta}} = m_0 - 2\mu_B a_{\vec{\delta}}^\dagger a_{\vec{\delta}}, \quad m_{x,\vec{\delta}} = (a_{\vec{\delta}} + a_{\vec{\delta}}^\dagger)\sqrt{m_0 \mu_B},$$

$$m_{y,\vec{\delta}} = i(a_{\vec{\delta}}^\dagger - a_{\vec{\delta}})\sqrt{m_0 \mu_B}; \quad m_{z,\vec{\mu}} = 2\mu_B b_{\vec{\mu}}^\dagger b_{\vec{\mu}} - m_0,$$

$$m_{x,\vec{\mu}} = (b_{\vec{\mu}} + b_{\vec{\mu}}^\dagger)\sqrt{m_0 \mu_B}, \quad m_{y,\vec{\mu}} = -i(b_{\vec{\mu}}^\dagger - b_{\vec{\mu}})\sqrt{m_0 \mu_B}. \tag{15}$$

Then the Hamiltonian takes the form, containing the terms describing the interaction of oscillations both within

first and second sublattices, as well as between different sublattices. The form of the corresponding terms is common to that for FM state, see Eq. (3), and we do not present this long expression here.

The translation symmetry within any sublattice allows us to introduce magnon creation and annihilation operators for magnetic oscillations through the Bose operators for each of the sublattices

$$a_{\vec{\delta}} = \frac{1}{\sqrt{N}} \sum_{\vec{q}} a_{\vec{q}} e^{ia\vec{q}\vec{\delta}}, \quad b_{\vec{\mu}} = \frac{1}{\sqrt{N}} \sum_{\vec{q}} b_{\vec{q}} e^{ia\vec{q}\vec{\mu}}, \tag{16}$$

where, naturally, the quasimomentum $\vec{q}$ takes values within the first Brillouin zone of the sublattice (see Fig. 4). In the basis of the sublattices the vector $\vec{q} = q_1 \vec{\varepsilon}_1 + q_2 \vec{\varepsilon}_2$, where $|q_{1,2}| \le \pi / a\sqrt{2}$, and in the basis of the basic lattice the vector $\vec{q} = q_x \vec{e}_x + q_y \vec{e}_y$, where $q_x = (q_1 + q_2)/\sqrt{2}$, $q_y = q_x = (q_2 - q_1)/\sqrt{2}$, and $|q_{x,y}| \le \pi / a$. After this transform, the Hamiltonian acquires the standard form $\hat{H} = \sum_{\vec{q}} \hat{H}_{\vec{q}}$, where

$$\frac{\hat{H}_{\vec{q}}}{2\mu_B M} = (A_{\vec{q}} + h) a_{\vec{q}}^\dagger a_{\vec{q}} + (A_{\vec{q}} - h) b_{\vec{q}}^\dagger b_{\vec{q}}$$

$$- \left[\frac{C_{\vec{q}}}{2}(a_{\vec{q}}^\dagger a_{-\vec{q}}^\dagger + b_{\vec{q}} b_{-\vec{q}}) + D_{\vec{q}} a_{\vec{q}}^\dagger b_{\vec{q}} + F_{\vec{q}} a_{\vec{q}} b_{-\vec{q}} + \text{H.c.} \right], \tag{17}$$
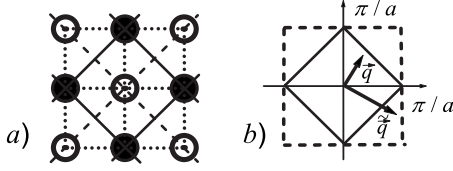
where the following notation are used:

FIG. 4. (a) The scheme of the sublattices for the particle array with the chessboard magnetic order; solid and dashed lines connect particles belonging to first and second sublattices, correspondingly; dotted lines connect particles from different sublattices. (b) The shape of the first Brillouin zone, the solid line restricts the Brillouin zone of the sublattice, the dashed line restricts the Brillouin zone for the whole lattice. The correspondence between $\vec{q}$ and $\widetilde{\vec{q}}$, introduced below, is also presented.

$$A_{\vec{q}} = \sum_{\vec{\mu}} \frac{1}{|\vec{\mu}|^3} - \sum_{\vec{\delta} \neq 0} \frac{1}{|\vec{\delta}|^3} - \frac{1}{2} \sum_{\vec{\delta} \neq 0} \frac{e^{ia\vec{q}\vec{\delta}}}{|\vec{\delta}|^3} + \beta,$$

$$C_{\vec{q}} = \frac{3}{2} \sum_{\vec{\delta} \neq 0} \frac{(\delta_x + i\delta_y)^2 e^{ia\vec{q}\vec{\delta}}}{|\vec{\delta}|^5},$$

$$D_{\vec{q}} = \frac{3}{2} \sum_{\vec{\mu}} \frac{(\mu_x + i\mu_y)^2 e^{ia\vec{q}\vec{\mu}}}{|\vec{\mu}|^5}, \quad F_{\vec{q}} = \frac{1}{2} \sum_{\vec{\mu}} \frac{e^{ia\vec{q}\vec{\mu}}}{|\vec{\mu}|^3}. \quad (18)$$

There is a simple connection between the sums over the sublattices and the previously introduced sums over the full lattice $\sigma(\vec{k})$ and $\sigma_c(\vec{k})$, see Eq. (8). For example, a simple geometrical transformation gives

$$\sum_{\vec{\delta} \neq 0} \frac{e^{ia\vec{q}\vec{\delta}}}{|\vec{\delta}|^3} = \frac{1}{2^{3/2}} \sigma(\widetilde{\vec{q}}),$$

where we introduced the vector $\widetilde{\vec{q}} = \sqrt{2}(q_1 \vec{e}_x + q_2 \vec{e}_y)$, which is derived from the vector $\vec{q}$ by rotation by the angle $\pi/4$ and by stretching by the value $\sqrt{2}$ (see Fig. 4). It is easy to see that for the $\vec{q}$ values within the first Brillouin zone of the sublattice, the corresponding values of $\widetilde{\vec{q}}$ are within the first Brillouin zone of the whole lattice. By using the rules of transition between the sums over the sublattices and the sum over the whole lattice one can obtain

$$A_{\vec{q}} = \left(1 - \frac{1}{\sqrt{2}}\right) \sigma(0) - \frac{1}{2^{5/2}} \sigma(\widetilde{\vec{q}}) + \beta,$$

$$C_{\vec{q}} = -\frac{3i}{2^{5/2}} \sigma_c(\widetilde{\vec{q}}), \quad D_{\vec{q}} = \frac{3}{2} \left[\sigma_c(\vec{q}) + \frac{i}{2^{3/2}} \sigma_c(\widetilde{\vec{q}}) \right],$$

$$F_{\vec{q}} = \frac{1}{2} \left[\sigma(\vec{q}) - \frac{1}{2^{3/2}} \sigma(\widetilde{\vec{q}}) \right]. \quad (19)$$

Thus, the Hamiltonian coefficients describing small oscillations of the AFM state do not include new dipole sums different from those for the FM case and could be expressed through the sums by means of cumbersome, but simple geometrical transformations. It is necessary to note that the Hamiltonian coefficients $F_{\vec{q}}$ and $D_{\vec{q}}$ in the terms such as $a_{\vec{q}} b_{-\vec{q}}$, $a_{\vec{q}} b_{\vec{q}}^\dagger$, responsible for interaction between sublattices, are periodic relative to vectors of the reciprocal lattice of the whole system, i.e., they have lower symmetry than the coefficients responsible for interaction within the sublattice. However, it can be proofed that in final expressions for collective mode frequencies the translation symmetry of the reciprocal lattice of period $\pi/a\sqrt{2}$ is restored.

B. Dispersion relation

For diagonalization of Hamiltonian (17) one can use the generalized Bogolyubov $u-v$ transformation and introduce creation and annihilation operators of magnons of two branches, $c_{\vec{q}}^\dagger$, $c_{\vec{q}}$ and $d_{\vec{q}}^\dagger$, $d_{\vec{q}}$, and the creation and annihilation operators of different branches commute, $[c_{\vec{q}}, d_{\vec{q}}] = 0$, $[c_{\vec{q}}, d_{\vec{q}}^\dagger] = 0$. For normal modes $\dot{c}_{\vec{q}} = -i\omega_{(-)}(\vec{q})c_{\vec{q}}$, $\dot{d}_{\vec{q}} = -i\omega_{(+)}(\vec{q})d_{\vec{q}}$, where $\omega_{(-)}(\vec{q})$ and $\omega_{(+)}(\vec{q})$ are the frequencies of magnon modes. The generalized Bogolyubov transform can be written as

$$a_{\vec{q}} = u_{\vec{q}} c_{\vec{q}} + v_{\vec{q}}^* c_{-\vec{q}}^\dagger + u_{\vec{q}}' d_{\vec{q}} + v_{\vec{q}}'^* d_{-\vec{q}}^\dagger,$$

$$b_{\vec{q}} = \xi_{\vec{q}} c_{\vec{q}} + \eta_{\vec{q}}^* c_{-\vec{q}}^\dagger + \xi_{\vec{q}}' d_{\vec{q}} + \eta_{\vec{q}}'^* d_{-\vec{q}}^\dagger. \quad (20)$$

Comparing the equations of motion for the operators $c_{\vec{q}}$ and $d_{\vec{q}}$ [for example, $\dot{c}_{\vec{q}} = -i\omega_{(-)}(\vec{q})c_{\vec{q}}$] and the operators $a_{\vec{q}}$ and $b_{\vec{q}}$ ($i\hbar\dot{a}_{\vec{q}} = [a_{\vec{q}}, \hat{H}]$), the system of equations for the coefficients is presented as a unitary transformation

$$\omega_{(\pm)}(\vec{q}) \begin{pmatrix} u_{\vec{q}} \\ v_{\vec{q}} \\ \xi_{\vec{q}} \\ \eta_{\vec{q}} \end{pmatrix} = \begin{pmatrix} (A_{\vec{q}} + h) & -C_{\vec{q}} & -D_{\vec{q}} & -F_{\vec{q}} \\ C_{\vec{q}}^* & -(A_{\vec{q}} + h) & F_{\vec{q}} & D_{\vec{q}}^* \\ -D_{\vec{q}}^* & -F_{\vec{q}} & (A_{\vec{q}} - h) & -C_{\vec{q}}^* \\ F_{\vec{q}} & D_{\vec{q}} & C_{\vec{q}} & -(A_{\vec{q}} - h) \end{pmatrix} \begin{pmatrix} u_{\vec{q}} \\ v_{\vec{q}} \\ \xi_{\vec{q}} \\ \eta_{\vec{q}} \end{pmatrix}. \quad (21)$$

In accordance with Eq. (21) the frequencies of collective modes of oscillations of dot magnetic moments are defined by a rather cumbersome formulae
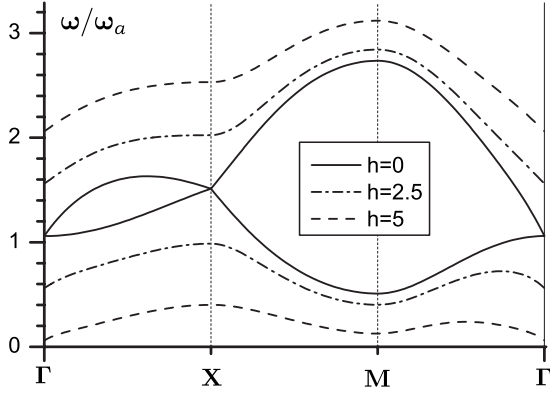
FIG. 5. The dispersion law along some symmetric directions for a dot array with a moderate anisotropy value (β=5) in the AFM state at different values of the magnetic field h (shown in the figure). The notations Γ, X, and M are the same as in Fig. 2, but for the Brillouin zone for sublattice.

$$\omega^2_{(\pm)}(\vec{q}) = (\gamma M)^2 [A^2 + |D|^2 + h^2 - F^2 - |C|^2$$
$$\pm 2\sqrt{h^2(A^2 - F^2) + |AD + FC|^2 - [\text{Im}(CD^*)]^2}] \tag{22}$$

but an analysis of the dispersion relation can be done numerically. The dependence of $\omega(\vec{q})$ for the two branches, $\omega_{(-)}(\vec{q})$ and $\omega_{(+)}(\vec{q})$, for specific values of a magnetic field are presented in Fig. 5.

The given expression is essentially simplified at symmetrical points of the Brillouin zone (in the zone center and on the edges), where the expression

$$\text{Im}(CD^*) = \frac{9}{2^{7/2}} \text{Re}[\sigma^*_c(\vec{q})\sigma_c(\tilde{\tilde{q}})]$$

becomes zero at these symmetrical point.

The spectrum behavior essentially depends on the external magnetic field value. The important and quite interesting (see below) case of small fields is determined by the behavior at h=0. Therefore, we begin with the study of asymptotics of the spectrum in the absence of an external magnetic field. Because of the long-range nature of the dipole interaction, the magnon spectrum has specific peculiarities in the center of the Brillouin zone, therefore it is reasonable to start with an analysis of the long-wave limit $k \to 0$.

To first order in $|\vec{k}|$ the sums take the form in Eq. (7) and after simple algebra one obtains the asymptotic form of the dispersion relation at h=0,

$$\omega_\pm(\vec{q}) = \gamma M \sqrt{\beta + (3\sqrt{2}/4)(\sqrt{2} - 1)\sigma(0)}$$
$$\times \sqrt{\beta - (1/2)(\sqrt{2} - 1)\sigma(0) + \pi a|\vec{q}|(1 \pm 1)}. \tag{23}$$

This expression describes a linear growth of frequency of the upper branch with increasing $q \equiv |\vec{q}|$ in the vicinity of the center of Brillouin zone. On the other hand, for the frequency of the lower branch $\omega_-(\vec{q})$ the terms linear in $\vec{q}$ are canceled out, and one can expect that for this branch the dependence on the components of the vector $\vec{q}$ should be parabolic. The numerical data presented in Fig. 5 indicate this. However, there is also a sharp anisotropy of the dependence $\omega_-(\vec{q})$, namely, in the vicinity of the point $\vec{q}$=0 the frequency $\omega_-(\vec{q})$ increases with a $q = |\vec{q}|$, if $\vec{q}$ is parallel to one of directions (1,0) and $\omega_-(\vec{q})$ decreases if $\vec{q}$ is parallel to (1,1). Note by virtue of fourth-order symmetry of the point $\vec{q}$=0 such behavior is forbidden for the quadratic terms. The anisotropy in the expansion of any analytical function $\omega(\vec{q})$ in the $\vec{q}$ components can appear only due to invariants of the fourth order, like $(q_x^4 + q_y^4)$ or $q_x^2 q_y^2$. Actually, again a nonanalyticity, caused by a slow convergence of the dipolar sums, takes place here. Our analysis shows in the vicinity of the point $\vec{q}$=0 the dispersion relation can be approximated by the expression

$$\omega(\vec{q}) \approx \omega_0 + \omega_0 a^2 \frac{\alpha(q_x^4 + q_y^4) - 2\beta q_x^2 q_y^2}{|\vec{q}|^2} \tag{24}$$

with the values of the coefficients, $\alpha \simeq 0.654$ and $\beta \simeq 1.8$. Thus, in the zero-field case the function $\omega(\vec{q})$ for the lower magnon branch has a new type of singular point, namely, the specific saddle point with fourfold symmetry.

In the case of nonzero external magnetic field in the vicinity of the point $q \to 0$ one gets the following expression at $a|\vec{q}| \ll h$ accurate to within first order in $|\vec{q}|$:

$$\omega_\pm(\vec{q}) = \gamma M \sqrt{\beta + (3\sqrt{2}/4)(\sqrt{2} - 1)\sigma(0)}$$
$$\times \sqrt{\beta - (1/2)(\sqrt{2} - 1)\sigma(0) + \pi a|\vec{q}|} \pm \gamma H. \tag{25}$$

The Eq. (25) shows that the frequencies of both branches of the spectrum in the vicinity of the center of the Brillouin zone increase linearly with $|\vec{q}|$ at a nonzero magnetic field.

C. Stability of AFM state

A study of the obtained dispersion relation allows a determination of an instability region for the AFM state, i.e., a region of parameters at which $\omega^2(\vec{q})$ becomes negative. The problem of finding the stability region for the AFM state is essentially simplified by the fact that the local minima of the lower branch of the magnon spectrum are located in the center ($\vec{q}_{(0,0)}$=0) or in the corners $[\vec{q}_{(1,1)} = \pi(\vec{\varepsilon}_1 + \vec{\varepsilon}_2)/(a\sqrt{2})]$ of the Brillouin zone of the sublattice, see Fig. 5. At these points some coefficients of the Hamiltonian vanish that makes a simple analytical consideration possible.

According to numerical calculations, in the absence of the external magnetic field and at small fields, the minimal value of frequency corresponds to magnons having the maximal value of quasimomentum, $\vec{q}_{(1,1)} = \pi(\vec{\varepsilon}_1 + \vec{\varepsilon}_2)/(a\sqrt{2}) = \pi\vec{\varepsilon}_x/a$. Also at this point $\omega_\pm(1,1) = A_{(1,1)} \pm \sqrt{D^2_{(1,1)} + h^2}$. An analysis of $\omega^2(\vec{q})$ in the vicinity of this point gives that at zero field the AFM state is stable only for enough large value of anisotropy, $\beta > \beta_{cr}$,
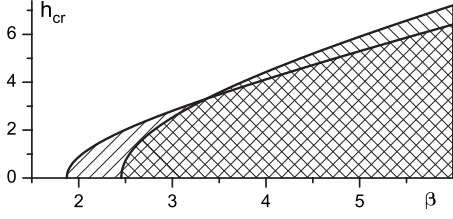
FIG. 6. The dependence of the critical magnetic field on anisotropy. Here both characteristic fields, $h_{cr,0}$ and $h_{cr,11}$, are plotted as functions of β, and the stability region of the AFM state is doubly patterned region under both curves.

$$\beta_{cr} = -\frac{3}{2}\sigma_c(\pi/a, 0) - \frac{7 - 3\sqrt{2}}{8}\sigma(0) \approx 2.4532. \quad (26)$$

If $\beta > \beta_{cr}$, then with increasing of the field the AFM state loses its stability relative to perturbations with $\vec{q}_{(1,1)}$ at $|h| > h_{cr,11} = \sqrt{A_{(1,1)}^2 - D_{(1,1)}^2}$, see. Fig. 5.

The analysis also shows that for $|h| > h_{cr,0} = \sqrt{A_0^2 - F_0^2}$ the AFM state loses the stability relative to small $\vec{q}$ perturbations. As it is seen from Fig. 5, both these conditions are important at different values of anisotropy. Thus, the AFM state is stable relative to arbitrary small perturbations subject to the condition, $h < h_{cr} = \min\{h_{cr,0}, h_{cr,11}\}$. For small anisotropy $\beta < \beta_1 = 3.358$ an instability with maximum values of q is developed whereas at $\beta > \beta_1$ the AFM state is unstable relative to long-wave perturbations with $q \ll 1/a$ (Fig. 6).

D. Density of states

Let us consider positions and form of Van Hove singularities for the spectral density of magnons of both branches as shown in Fig. 7. In a weak field, frequency bands corresponding to upper and lower branches of collective oscillations overlap. In this case it is convenient to introduce the partial density of states as shown in Fig. 7(a) for zero field. As for the aforementioned FM state of the array nonstandard behavior of $D(\omega)$ for both branches can merge caused by long-range character of interaction. As the previously discussed example, when the discontinuity of the derivative of density of states has the form $D(\omega) = C_1 + C_2(\omega - \omega_0)\Theta(\omega - \omega_0)$, this corresponds to the linear dependence of $\omega(\vec{q})$ in the vicinity of the center of the Brillouin zone, $\omega(\vec{q}) - \omega_0 \propto |\vec{q}|$. For the AFM case at various values of magnetic field this possibility can be realized, but the situation can be richer compared to the FM case. In particular, this peculiarity is manifested differently in the density of states for upper and lower bands.

It happens that the standard scheme of extrema [a minimum in the center of the Brillouin zone, maxima on the edges of the Brillouin zone, saddle points in four symmetrical points like X(1,0)] takes place only for the upper branch and only at large enough field. In this case in the upper band there are only three Van Hove singularities [see Figs. 7(b) and 7(c)]. For the upper branch the point $\vec{q} = 0$ at any field always defines the global minimum and peculiarities like $D(\omega) \propto (\omega - \omega_0)\Theta(\omega - \omega_0)$ define the density of states near the lower edge of the upper band. For all fields the absolute

maximum of the function $\omega_+(\vec{k})$ having a parabolic dependence is located at the corners of the Brillouin zone at points such as M(1,1). For this reason, one expects the standard peculiarity of $D(\omega)$ like a finite discontinuity near the maximal frequency. For the upper band a nonstandard behavior manifests itself only at small fields $h < 1.2$, when at points like X(1,0) minima appear, while saddle points move along directions like (1,0) into the Brillouin zone. The situation here resembles that described for the FM case and we do not discuss it here.

The scheme of Van Hove singularities for the lower branch are much less standard. First of all, for the low band at all values of field the global maximum is located at the points like X(1,0). A standard peculiarity of the density of states like a finite jump corresponds to this frequency. At $h \neq 0$ (the zero-field case is special and will be considered separately) both at the edges of the Brillouin zone [points like M(1,1)] and in its center the minima are present, and the relative depth of the minima at points Γ and M is defined by a field value. Thus, all symmetrical points of the Brillouin zone, Γ, X, and M are occupied by minima or maxima. For this reason the saddle points, to which standard logarithmic peculiarities of density of states correspond, are moved inside the Brillouin zone along a direction like (1,0).

The relative position of minima is defined by a value of the magnetic field. For the lower branch at weak fields ($h < 3$) the absolute minimum of the function $\omega_-(\vec{k})$ is located at a point like M. The parabolic dependence $\omega(\vec{q})$ corresponds to this minimum, and the latter defines a standard peculiarity of the density of states like a finite discontinuity [see Figs. 7(a) and 7(b)]. At such values of the field the peculiarity connected with the point $\Gamma(\vec{q} = 0)$ lies within the lower band frequency and manifests itself as a derivative jump $D(\omega)$ (for example, at $\omega = 0.6\omega_a$ for $h = 2.5$) and it is almost invisible in the figures. For higher fields ($h > 3$) the frequency minimum at the point $\Gamma(\vec{q} = 0)$ becomes deeper, i.e., the local minimum value of the lower branch of the spectrum [$\omega_-(M)$] lies higher than the frequency of long-wave oscillations $\omega_-(0)$, (see Fig. 7). In this case a linear behavior of the density of states $D(\omega) \sim (\omega - \omega_0)\Theta(\omega - \omega_0)$ is clearly seen at the lower edge of the frequency band for the lower branch. In this case, within the band there is one more singularity of the density of states, which is a finite jump of the density of states, determined by a local parabolic minimum at the point M(1,1), see Fig. 7(c).

One of the most interesting singularities of the spectrum of the lower branch is observed in the special case of zero field. First, in this case the global maxima at the points X(1,0) have a nonparabolic dependence $\omega(\vec{q})$, $\omega(\vec{q}) = \omega_{max} - c|\Delta\vec{q}|$, where $\Delta\vec{q}$ is a deviation of $\vec{q}$ from the point X. For this reason, the upper edge of the energy band is characterized be nonstandard linear Van Hove singularity, $D(\omega) = C(\omega_{max} - \omega) \cdot \Theta(\omega_{max} - \omega)$. On the other hand in the expansion of $\omega(\vec{q})$ in the center of the Brillouin zone the linear term in q is absent, and there is a specific saddle point with fourfold symmetry having a nonanalytic behavior, see Eq. (24). However the analysis shows that this saddle point leads to the standard logarithmic Van Hove singularity. For small, but finite value of the external field in the zone center there is
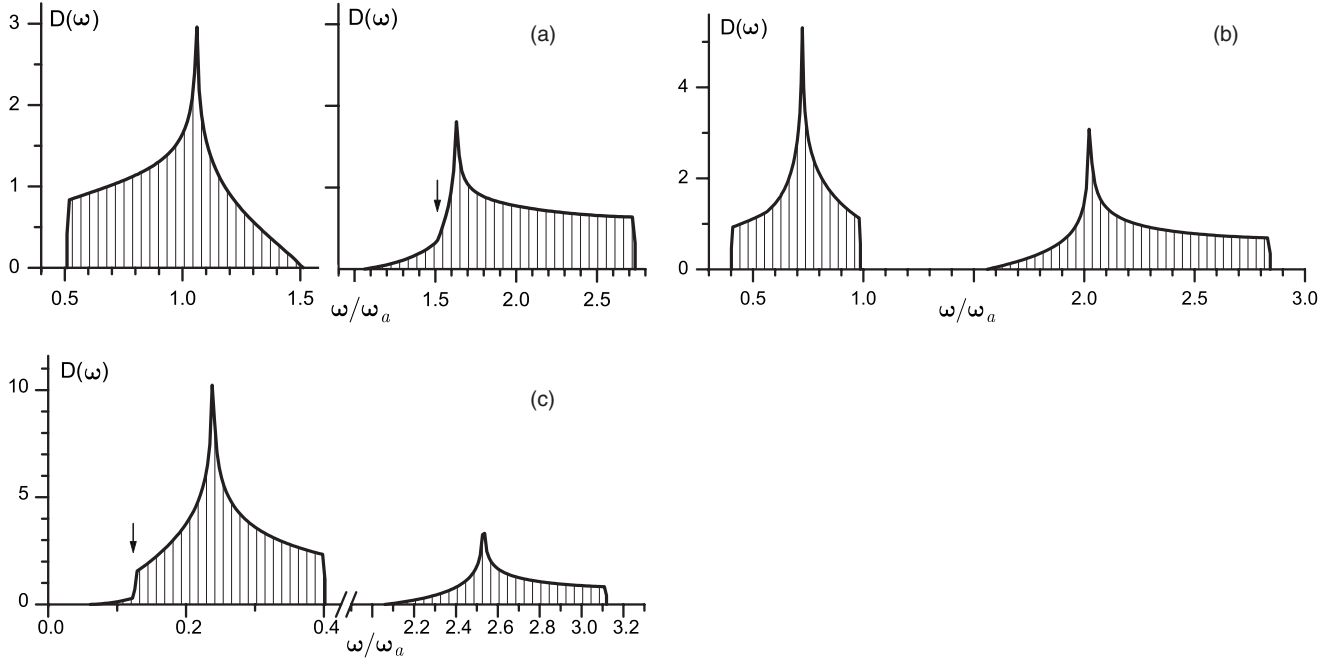
FIG. 7. The density of state for two branches of magnons for the AFM dot array with $\beta=5$, normalized by the condition $\int D(\omega)d\omega/\omega_a=1$, $\omega_a=\gamma H_a$, for each band, for different values of the magnetic field. (a) The field equals zero, pay attention that these two branches are overlapped. (b) $h=2.5$, at moderate field values a linear peculiarity of the density of states is present inside of the lower band of collective modes. (c) $h=5$ presents the case of large fields, where a linear peculiarity of the density of states corresponds to lower frequency, and the finite jump is present inside the low-energy band. The characteristic frequencies are mentioned by vertical arrows.

a linear nonanalytical behavior of $\omega(\vec{q})\approx\omega_0+\alpha|\vec{q}|$, which is typical for the systems having dipole interaction. While at the point $\vec{q}=0$ there is a minimum and a nonanalytic saddle point splits into four standard saddle points moved toward the points X.

V. CONCLUSION

First let us discuss one possible generalization of our approach for the problem of interest not considered here. The concrete calculations here have been done for the most symmetric configurations of the system, namely, the ideal square-lattice array and individual dots having rotational uniaxial symmetry. The consideration of the same problem with lower symmetry will not produce any principal difficulties. For a lattice of lower symmetry, such as rectangular, the same peculiarities for dipolar sums and long-wave asymptotics of dispersion relations are present. The case of dots of lower symmetry, such as rectangular or elliptic, with almost homogeneous magnetization within the dots can be easily analyzed, just by adding to Eq. (1) extra terms describing more complicated (biaxial) anisotropy of a single dot. Qualitatively the results will be the same.

Strictly speaking, our theory based on the model of macrospins is valid only for arrays of small magnetic elements of the shape of ellipsoid. However we believe, it also can be applied, at least qualitatively, to arrays of small enough non-ellipsoidal elements where the distribution of magnetization is almost uniform. On the other hand, the theory developed here obviously is not adequate to description of so-called edge modes, present at micron-sized thin rectangular mag-

netic dots, and for other magnetic elements with strictly non-uniform modes. For description of this interesting case, an essential modification of the theory should be developed that is far beyond the scope of our paper.

In conclusion, the dynamic properties of the array of magnetic dots of different types without direct exchange interaction between dots have been investigated. For such systems, only the magnetic-dipole interaction can be a source of dot interaction. Exploiting the translation symmetry of the array, and by use of the quasiparticle (magnon) formalism, we have found full spectra for the change of quasimomentum within all of the Brillouin zone. The direct measurement of the dependence $\omega(\vec{k})$ can be done by the Brillouin light-scattering method. Our calculations demonstrate that the dispersion is a strongly increasing function of the dot thickness. It will be interesting to observe nonmonotonic dependence $\omega(\vec{k})$ and especially, the change of the character of this dependence with the change in the magnetic field found here. The most impressive change in the $\omega(\vec{k})$ dependence should appears near the transition from FM to AFM states of an array. This property can be of interest for the design of a new generation of microwave devices in the modern direction of the applied physics of magnetic nanoparticles, i.e., the so-called magnonics, which has been widely discussed in the literature.[43]

An important application of the calculated dispersion laws is the investigation of the stability of given magnetic states of the array. It has been found that for arrays with homogeneous magnetization within a dot directed perpendicular to the array's plane there will always be a nonzero critical field, either external or in combination with an anisotropy field. For ferromagnetic state of array, the instability is developed

for the quasimomentum at the points of the type of (1,0) at the border of the Brillouin zone, and it leads to a transition to the AFM state with chessboard orientation of the dot's magnetic moment.

For all magnetic states of the array investigated here, the dispersion relation is nonanalytic as $\vec{k} \to 0$ because of the long-range nature of the dipolar interaction of oscillating magnetic moments of dots and the presence of singularities in dipolar sums. For such modes with finite-gap frequency, ω_0 the magnon spectra have the peculiarity of the type $\omega(\vec{k}) - \omega_0 \propto |\vec{k}|$ as $\vec{k} \to 0$. For the AFM structure without a magnetic field, a new sort of singularity, leading to saddle point with fourfold symmetry of the function $\omega(\vec{k})$ near the value $\vec{k} = 0$ is predicted. An important and nontrivial property of most of the modes considered here is the strong anisotropy of the function $\omega(\vec{k})$. Note also the decreasing or nonmonotonic dependence of the mode frequency $\omega(\vec{k})$ on the wave vector $\vec{k}$, observed for nonsmall interaction of dots.

The nonstandard dependence $\omega(\vec{k})$ for small k produce the change of the density of states $D(\omega)$. For the spectrum of the FM state, instead of standard two-dimensional Van Hove singularities of a form of finite jump near the gap frequency, ω_0, or singularities like $\ln(\omega_c / |\omega_c - \omega|)$, weaker singularities of the form $D(\omega) \sim (\omega - \omega_0) \Theta(\omega - \omega_0)$ appear here. On the other hand, for dispersion relations for the FM state [Eq. (11)] the change of the character of its extremum at the point X of the type of (1,0) is predicted at some ratio of the system parameters, $\omega_0 / \omega_{int} \simeq 3.6$ (see Fig. 2). For this critical value of the parameter ω_0 / ω_{int} the density of states has the singularity $D(\omega) \propto 1/|\omega - \omega_c|^{1/4}$, where ω_c is the value of frequency at this critical point, which is stronger than the standard finite jump. The value of the parameter ω_0 / ω_{int} depends on the external magnetic field and can be changed continuously during the experiment.

APPENDIX

The expressions for collective-mode frequencies contain series such as the dipole sums $\sigma(\vec{k})$ and $\sigma_c(\vec{k})$. Here and below in this section the condition $\vec{l} \neq 0$ in the sums is implied. The mathematical properties of such series are important not only for this problem but also for any example of lattice systems with identical particles coupled by the dipole interaction.

Let us discuss properties of these series. As we will demonstrate these series have very singular behavior as a functions of quasimomentum $\vec{k}$, and the double sum converges rather slowly. This manifests in sum properties near symmetrical points of the reciprocal lattice $\vec{k}_0$, especially near the point of origin $\vec{k}_0 = 0$ or near the points of the type of M(1,1),

for which $\vec{k}_0 = \pm(\vec{e}_x \pm \vec{e}_y)\pi/a$, and X(1,1), for which $\vec{k}_0 = \pm(\vec{e}_x)\pi/a$ or $\vec{k}_0 = (\pm\vec{e}_y)\pi/a$. Analyzing small deviations from these points, $\vec{k} = \vec{k}_0 + \vec{q}$, where $\vec{q}$ is small, one normally has to calculate derivatives such as $[\partial^2\sigma(\vec{k})/\partial q_i \partial q_j]$ at the point $\vec{k} = \vec{k}_0$. Term by term differentiation of the dipole sums gives series like $\Sigma(1/|\vec{l}|)\exp(i\vec{k}_0\vec{l})$, which are alternating and converge only conditionally. Moreover, for $\vec{k}_0 = 0$, i.e., for the physically most interesting case of long-wave oscillations, the corresponding coefficient of $\vec{q}^2$ is described by the divergent series $\Sigma 1/|\vec{l}|$. The same property is present for the complex sum $\sigma_c(\vec{k})$, which is also important for the description of dipole-coupled modes. Therefore, the dipole sums at $\vec{k} \simeq 0$ can be nonanalytical and it results in a nonstandard dispersion relation for oscillations described above.

Let us start with study of the sum $\sigma(\vec{k})$ for small $|\vec{k}|/k_B$. To analyze the behavior of $\sigma(\vec{k})$ near the point $\vec{k} = 0$ one can write $\vec{k} = \vec{q}$, $|\vec{q}| \ll 1/a$ and present this series as

$$\sum \frac{e^{i\vec{q}\vec{l}}}{|\vec{l}|^3} = \sum \frac{1}{|\vec{l}|^3}[e^{i\vec{q}\vec{l}} \cdot e^{-\alpha l^2} + e^{i\vec{q}\vec{l}} \cdot (1 - e^{-\alpha l^2})], \quad \text{(A1)}$$

where the multiplier $\exp(-\alpha|\vec{l}|^2)$ is chosen to provide a fast convergence of the corresponding series. The first term on the right-hand side of Eq. (A1) converges rapidly for $|\vec{l}| > 1/\sqrt{\alpha}$ and at $q = |\vec{q}| \to 0$ it contributes as $-q^2 D'$, where $D' = \Sigma(1/|\vec{l}|)\exp(-\alpha l^2)$ is an absolutely converging series. Nonanalyticity as $q \to 0$ is defined by the second term, calculation of which can be simplified. Indeed, at $|\vec{l}| \ll 1/\sqrt{\alpha}$ it comprises the small multiplier αl^2, and the contribution from the region $|\vec{l}| < 1/\sqrt{\alpha}$ is expected to be small. For the outer region $|\vec{l}| > 1/\sqrt{\alpha}$ one can expect that discreteness effects are small and the sum can be replaced by the integral, $\Sigma_{\vec{l}} \to \int r dr d\chi$, where r and χ are the polar coordinates. Integration over χ can be done exactly through the Bessel function $J_0(qr)$ and the nonanalytical part of the sum $\sigma(\vec{k})$ at small $|\vec{k}|$ is determined by the integral $I = 2\pi\int_0^\infty (dr/r^2)J_0(qr)[1 - \exp(-\alpha r^2)]$, which is convergent both as $r \to 0$ and $r \to \infty$. It is convenient to integrate by parts, which in the case we are interested in $(\alpha \to 0)$ yields $I = 2\pi\int_0^\infty (dr/r)J_1(qr) = 2\pi|q|$. From here we arrive at the expression, used in Eq. (7)

$$\sigma(\vec{k}) = \sigma(0) - 2\pi a|\vec{k}| \text{ at } \vec{k} \to 0.$$

For the complex sum $\sigma_c(\vec{k})$ near the origin $\vec{k} = \vec{q}$, $|\vec{q}| \ll 1/a$ the same approach gives the double integral,

$$\sigma_c(\vec{q}) = \int_0^\infty \frac{dr}{r^2} \cdot \int_0^{2\pi} d\chi e^{-2i\chi} \cdot e^{iqr\cos(\chi-\alpha)},$$

where α is the angle between $\vec{q}$ and the x axis. Here the integral over χ can be done, which gives $\sigma_c = 2\pi e^{-2i\alpha}\int(dr/r^2)J_2(qr)$. Further, for the integral over r, which is convergent both as $r \to 0$ due to asymptotic $J_2(z) \to (1/2)(z/2)^2$ as $z \to 0$ and for $r \to \infty$, it is not necessary to include the regularization multiplier $\exp(-\alpha r^2)$ like in Eq. (A1). Finally we arrive at the following expression:

$$\sigma_c(\vec{k}) = \frac{2\pi}{3} a \frac{(k_x - ik_y)^2}{|\vec{k}|} \quad \text{as} \quad \vec{k} \to 0.$$

Thus the relation between $\sigma(\vec{k})$ and $\sigma_c(\vec{k})$, $\sigma(\vec{k}) \to 3|\sigma_c(\vec{k})|$ as $\vec{k} \to 0$, which is of great importance for the spectral analysis is asymptotically exact at small values of $|\vec{k}|$. This result is in accordance with the numerical calculation.

Let us discuss now the calculation of the series $\sigma(\vec{k})$ and $\sigma_c(\vec{k})$ near the second symmetric point $\vec{k}_0$ of the type of (1,1). Here for both series the values of $D_{ij} = [\partial^2 \sigma(\vec{k})/\partial k_i \partial k_j]_{\vec{k}=\vec{k}_0}$ are described by convergent alternating series like $\sum P_{2p}(\vec{l}) \cdot (-1)^{l_1 + l_2}(l_1^2 + l_2^2)^{-(2p+1)/2}$, where $P_{2p}(\vec{l})$ is the polynomial in l_x, l_y of degree $2p$, $p = 1$ and $p = 2$ for components D_{ij} in the case of $\sigma(\vec{k})$ and $\sigma_c(\vec{k})$, respectively.

By using symmetry relations, one can demonstrate that the value of the function $\sigma(\vec{k})$ at $\vec{k}$ of type (1,1) is described only by one sum, $D = -\sum (-1)^{l_1 + l_2}(l_1^2 + l_2^2)^{-1/2}$, the value of which can be easily found numerically, $D \cong 1.61554$. As a result the following expression is obtained:

$$\sigma(\vec{k}) \cong \sigma(\vec{k}_0) + \frac{1}{4} D a^2 |\vec{k} - \vec{k}_0|^2.$$

For analysis of the sum $\sigma_c(\vec{k})$ near the point $\vec{k} = \vec{k}_0$ we note that because of the evident relations

$$\sum_{\vec{l} \neq 0} (l_1^2 - l_2^2)(l_1^2 + l_2^2)^{-5/2}(-1)^{l_1 + l_2} = 0,$$

the value of $\sigma_c(\vec{k}_0) = 0$. The next terms of the expansion $\sigma_c(\vec{k})$ over the vector components $\vec{q} = \vec{k} - \vec{k}_0$ can be written easily,

$$\sigma_c(\vec{k}) = 2i q_x q_y \sum \frac{l_1^2 l_2^2}{(l_1^2 + l_2^2)^{5/2}} (-1)^{l_1 + l_2}$$
$$+ \frac{1}{2}(q_x^2 - q_y^2) \sum \frac{l_2^2(l_1^2 - l_2^2)}{(l_1^2 + l_2^2)^{5/2}} (-1)^{l_1 + l_2}. \quad \text{(A2)}$$

They comprise both real terms proportional to $q_x^2 - q_y^2$ and imaginary terms like $q_x q_y$. After simple algebra, these terms are presented via the sum D introduced above and one more convergent sum of similar structure

$$D_1 = -\sum (l_1^2 - l_2^2)^2 (-1)^{l_1 + l_2}/(l_1^2 + l_2^2)^{-5/2} \cong 3.31891.$$

Finally, in the quadratic approximation over $\vec{q} = \vec{k} - \vec{k}_0$ near the point $\vec{k}_0$ of the type (1,1) one can present the sum as

$$\sigma_c(\vec{k}) = \sigma_c(\vec{k}_0) + i(D_1 - D) a^2 q_x q_y + \frac{1}{4} D_1 a^2 (q_x^2 - q_y^2).$$

This function, in contrast with $\sigma(\vec{k})$ near the same point $\vec{k}_0$, is anisotropic. However, expressions for the frequencies contain $\sigma(\vec{k})$ and $\sigma_c(\vec{k})$ only in the combination $\sigma^2(\vec{k}) - 9|\sigma_c(\vec{k})|^2$, therefore for presentation of the spectrum of collective modes accurate to $|\vec{q}|^2$ it is not essential to take $\sigma_c(\vec{k})$ into account, and to the accuracy of q^2 the spectrum near the point $\vec{k}_0$ of the type (1,1) is radially symmetric over components of the vector $\vec{q}$.

The point with $\vec{k}_0$ of the type (1,0) including the points like $\vec{k}_0 = \pm \pi \vec{e}_x / a$ or $\vec{k}_0 = \pm \pi \vec{e}_y / a$ possess lower (biaxial) symmetry than the points $\vec{k} = 0$ or points of the type (1,1). The rather complicated study in the vicinity of this point ($\vec{q} = \vec{k} - \vec{k}_0$ is small) shows the analytical dependence for the components of $\vec{q}$, in this case for the components of $\vec{q}$ parallel and perpendicular to $\vec{k}_0$

$$\sigma(\vec{k}) - \sigma(\vec{k}_0) = \frac{1}{2} a^2 (D_\parallel q_\parallel^2 - D_\perp q_\perp^2),$$

where $D_\parallel$ and $D_\perp$ are convergent series of the form

$$D_\parallel = -\sum \frac{l_1^2(-1)^{l_1}}{(l_1^2 + l_2^2)^{5/2}} \cong 0.6354,$$

$$D_\perp = \sum_{\vec{l} \neq 0} \frac{l_2^2(-1)^{l_1}}{(l_1^2 + l_2^2)^{5/2}} \cong 1.256. \quad \text{(A3)}$$

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